

# New Compactifications of Eleven Dimensional Supergravity

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## Abstract

Using canonical forms on  $S^7$ , viewed as an  $SU(2)$  bundle over  $S^4$ , we introduce consistent ansätze for the 4-form field strength of eleven-dimensional supergravity and rederive the known squashed, stretched, and the Englert solutions. Further, by rewriting the metric of  $S^7$  as a  $U(1)$  bundle over  $\mathbf{CP}^3$ , we present yet more general ansätze. As a result, we find a new compactifying solution of the type  $AdS_5 \times \mathbf{CP}^3$ , where  $\mathbf{CP}^3$  is stretched along its  $S^2$  fiber. We also find a new solution of  $AdS_2 \times H^2 \times S^7$  type in Euclidean space.

# 1 Introduction

Eleven-dimensional supergravity solutions have been extensively studied in 1980's. Among these, the Freund-Rubin solution [1] was the simplest one as it included a 4-form field strength with components only along the  $AdS$  direction. Then, attentions were turned to possible solutions with nonvanishing components along the compact directions. Englert was the first to construct such a solution;  $AdS_4 \times S^7$  with the round metric on  $S^7$  [2]. Later, the so-called squashed solutions with non-standard Einstein metric on  $S^7$  were found [3, 4]. Here,  $S^7$  is considered as an  $SU(2)$  bundle over  $S^4$  and squashing corresponds to rescaling the metric along the fiber. For a specific value of the squashing parameter the metric turns out to be Einstein.

In constructing the Englert type solutions Killing spinors play a significant role. Killing spinors are also required for having supersymmetric solutions [5, 7, 8, 9, 10]. Alternatively, on compact manifolds with a bundle structure on a Kähler base, one can use the holomorphic top form and the Kähler form to write consistent ansätze for the 4-form field strength [11]. Algebraic approaches have also been used to study the supergravity solutions [12].

In the present work, however, instead of looking for Killing spinors we directly use canonical forms on  $S^7$  to write a consistent ansatz for the 4-form field strength. In particular, this allows us to rederive the squashed, stretched, and the Englert solutions in a unified scheme. There are some independent earlier works which also use canonical geometric methods [13, 14]. In Sec. 2, we consider  $S^7$  as an  $S^3$  bundle over  $S^4$ , and identify a natural basis of such forms in terms of the volume forms of the fiber and the base. We will see that a linear combination of these forms provides a suitable ansatz for the Maxwell equation, so that the field equations reduce to algebraic equations for the parameters of the ansatz. In Sec. 3, we rewrite the squashed metric of  $S^7$  as a  $U(1)$  bundle over  $\mathbf{CP}^3$ , where it appears as an  $S^2$  bundle over  $S^4$  with rescaled fibers. Moreover, in this form, we can introduce a different rescaling parameter for the  $U(1)$  fibers. This enables us to provide more general ansätze. In Sec. 4, we consider a direct product of a 5 and 6-dimensional spaces and find a new compactifying solution of  $AdS_5 \times \mathbf{CP}^3$ , in which  $\mathbf{CP}^3$  is stretched along its  $S^2$  fiber. In Sec. 5, we discuss solutions in which the eleven dimensional space has a Euclidean signature and is a direct product of two 2-dimensional spaces and  $S^7$ . We find a solution of  $AdS_2 \times H^2 \times S^7$  type, in which  $H^2$  is a hyperbolic surface, and  $S^7$  is stretched along its  $U(1)$  fiber by a factor of 2.

## 2 Squashed solution revisited

Let us start our discussion with the Freund-Rubin solution, for which the 4-form field strength has components only along the four dimensions

$$F_4 = \frac{3}{8} R^3 \epsilon_4, \quad (1)$$

and the metric reads

$$ds^2 = R^2 \left( \frac{1}{4} ds_{AdS_4}^2 + ds_{S^7}^2 \right). \quad (2)$$

The round metric on  $S^7$  can be written as an  $SU(2)$  bundle over  $S^4$  [3, 15]

$$ds_{S^7}^2 = \frac{1}{4} (d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + (\sigma_i - \cos^2 \mu / 2 \Sigma_i)^2), \quad (3)$$

with  $0 \leq \mu \leq \pi$ , and  $\Sigma_i$ 's and  $\sigma_i$ 's are two sets of left-invariant one-forms

$$\begin{aligned} \Sigma_1 &= \cos \gamma d\alpha + \sin \gamma \sin \alpha d\beta, \\ \Sigma_2 &= -\sin \gamma d\alpha + \cos \gamma \sin \alpha d\beta, \\ \Sigma_3 &= d\gamma + \cos \alpha d\beta, \end{aligned}$$

where  $0 \leq \gamma \leq 4\pi$ ,  $0 \leq \alpha \leq \pi$ ,  $0 \leq \beta \leq 2\pi$ , and with a similar expression for  $\sigma_i$ 's. They satisfy the  $SU(2)$  algebra

$$d\Sigma_i = -\frac{1}{2} \epsilon_{ijk} \Sigma_j \wedge \Sigma_k, \quad d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k, \quad (4)$$

with  $i, j, k, \dots = 1, 2, 3$ .

Squashing corresponds to modifying the round metric on  $S^7$  as follows

$$ds_{S^7}^2 = \frac{1}{4} (d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (\sigma_i - \cos^2 \mu / 2 \Sigma_i)^2), \quad (5)$$

with  $\lambda$  the squashing parameter. So, let us take the following ansatz for the 11d metric:

$$ds^2 = \frac{R^2}{4} \left( ds_4^2 + d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (\sigma_i - \cos^2 \mu / 2 \Sigma_i)^2 \right), \quad (6)$$

and choose the orthonormal basis of vielbeins as

$$e^0 = d\mu, \quad e^i = \frac{1}{2} \sin \mu \Sigma_i, \quad \hat{e}^i = \lambda (\sigma_i - \cos^2 \mu / 2 \Sigma_i). \quad (7)$$

Further, in order to construct our ansatz in the next section we need to evaluate the exterior derivatives of the vielbeins

$$de^i = \cot \mu e^0 \wedge e^i - \frac{1}{\sin \mu} \epsilon_{ijk} e^j \wedge e^k \quad (8)$$

$$d\hat{e}^i = \lambda e^0 \wedge e^i + \frac{1}{2} \epsilon_{ijk} \left( \lambda e^j \wedge e^k - \frac{1}{\lambda} \hat{e}^j \wedge \hat{e}^k - 2 \left( \frac{1 + \cos \mu}{\sin \mu} \right) e^j \wedge \hat{e}^k \right), \quad (9)$$

where use has been made of (4).

## 2.1 The ansatz

Let us now introduce  $\omega_3$ , the volume element of the fiber  $S^3$ :

$$\omega_3 = \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3, \quad (10)$$

taking the derivative along with using (9), we obtain

$$d\omega_3 = \frac{\lambda}{2} \left( \epsilon_{ijk} e^0 \wedge e^i \wedge \hat{e}^j \wedge \hat{e}^k + e^i \wedge e^j \wedge \hat{e}^i \wedge \hat{e}^j \right). \quad (11)$$

The Hodge dual reads

$$*d\omega_3 = \lambda \hat{e}^i \wedge (e^0 \wedge e^i + \frac{1}{2} \epsilon_{ijk} e^j \wedge e^k), \quad (12)$$

so that using (8) and (9), we derive

$$d * d\omega_3 = 6\lambda^2 \omega_4 - \frac{1}{\lambda} d\omega_3, \quad (13)$$

where

$$\omega_4 = e^0 \wedge e^1 \wedge e^2 \wedge e^3, \quad (14)$$

is the volume element of the base. Note that  $\omega_4$  is closed;  $d\omega_4 = 0$ . Further, since  $d * \omega_4 = d\omega_3$ , for a linear combination of these two forms we have

$$d * (\alpha \omega_4 + \beta d\omega_3) = 6\lambda^2 \beta \omega_4 + (\alpha - \beta/\lambda) d\omega_3 \quad (15)$$

namely, the subspace with a basis of  $\omega_4$  and  $d\omega_3$  is closed under  $d*$  operation. This is exactly what we need to construct a consistent ansatz for the 4-form field strength.

The above analysis shows that we can take the following ansatz:

$$F_4 = N\epsilon_4 + \alpha \omega_4 + \beta d\omega_3, \quad (16)$$

with  $N$ ,  $\alpha$ , and  $\beta$  constant parameters to be determined by field equations, also note that  $dF_4 = 0$ . Substituting this into the field equation<sup>1</sup>

$$d *_{11} F_4 = -\frac{1}{2} F_4 \wedge F_4, \quad (17)$$

we get

$$\frac{R^3}{8} d(N \omega_3 \wedge \omega_4 + \alpha \epsilon_4 \wedge \omega_3 + \beta \epsilon_4 \wedge * d\omega_3) = -N \epsilon_4 \wedge (\alpha \omega_4 + \beta d\omega_3), \quad (18)$$

therefore, using (13), we must have

$$6\lambda^2 \beta = -\frac{8N}{R^3} \alpha, \quad \alpha - \frac{\beta}{\lambda} = -\frac{8N}{R^3} \beta. \quad (19)$$

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<sup>1</sup>The star in this equation is the eleven-dimensional Hodge dual operation. In the rest of equations, it indicates the seven dimensional Hodge dual operation.

A nontrivial solution exists if

$$\lambda \left( \frac{8N}{R^3} \right)^2 - \frac{8N}{R^3} - 6\lambda^3 = 0. \quad (20)$$

We will return to this equation after discussing the Einstein equations.

Now let us turn to the Einstein equations:

$$R_{MN} = \frac{1}{12} F_{MPQR} F_N{}^{PQR} - \frac{1}{3 \cdot 48} g_{MN} F_{PQRS} F^{PQRS}, \quad (21)$$

where  $M, N, P, \dots = 0, 1, \dots, 10$ . With ansatz (16), we can calculate the right hand side of the above equations:

$$R_{\mu\nu} = \left( \frac{4}{R^2} \right)^4 \left( -\frac{3!}{12} N^2 - \frac{4!}{3 \cdot 48} (-N^2 + \alpha^2 + 6\lambda^2 \beta^2) \right) g_{\mu\nu}, \quad (22)$$

$$R_{\alpha\beta} = \left( \frac{4}{R^2} \right)^4 \left( \frac{3!}{12} (\alpha^2 + 3\lambda^2 \beta^2) - \frac{4!}{3 \cdot 48} (-N^2 + \alpha^2 + 6\lambda^2 \beta^2) \right) \delta_{\alpha\beta}, \quad (23)$$

$$R_{\hat{\alpha}\hat{\beta}} = \left( \frac{4}{R^2} \right)^4 \left( \frac{3!}{12} (4\lambda^2 \beta^2) - \frac{4!}{3 \cdot 48} (-N^2 + \alpha^2 + 6\lambda^2 \beta^2) \right) \delta_{\hat{\alpha}\hat{\beta}}, \quad (24)$$

with  $\mu, \nu = 0, \dots, 3$ ,  $\alpha, \beta = 4, \dots, 7$ , and  $\hat{\alpha}, \hat{\beta} = 8, 9, 10$ . Notice that different terms in our ansatz (16) do not contract into each other. For the left hand side, on the other hand, the Ricci tensor of metric (6) becomes

$$\begin{aligned} R_{\alpha\beta} &= \left( \frac{4}{R^2} \right) \left( \frac{3(2 - \lambda^2)}{2} \right) \delta_{\alpha\beta}, \\ R_{\hat{\alpha}\hat{\beta}} &= \left( \frac{4}{R^2} \right) \left( \frac{1 + 2\lambda^4}{2\lambda^2} \right) \delta_{\hat{\alpha}\hat{\beta}}, \end{aligned} \quad (25)$$

these are to be substituted on the left hand side of (23) and (24).

We can now solve (19) and (20) for  $\beta$  and  $N$ , and then plug it into (23) and (24). The two resulting equations can be solved for  $\lambda$  and  $\alpha$ . We get two types of solutions. Those with no internal flux:

$$\alpha = \beta = 0, \quad (26)$$

together with  $\lambda^2 = 1$ , which is the round sphere. Or, we can have  $\lambda^2 = 1/5$ , which corresponds to the squashed sphere solution. We also get solutions with fluxes

$$\alpha^2 = 9/5, \quad \beta^2 = 9, \quad \lambda^2 = 1/5. \quad (27)$$

For  $\lambda = 1/\sqrt{5}$ ,  $\alpha = -3/\sqrt{5}$ ,  $\beta = 3$ , and  $N = 3R^3/(4\sqrt{5})$  we have a non-zero 4-form field strength along  $S^7$ , and it represents the squashed  $S^7$  with Einstein metric

$$R_{\alpha\beta} = \left( \frac{4}{R^2} \right) \frac{27}{10} \delta_{\alpha\beta}, \quad R_{\hat{\alpha}\hat{\beta}} = \left( \frac{4}{R^2} \right) \frac{27}{10} \delta_{\hat{\alpha}\hat{\beta}}. \quad (28)$$

The above solution, the so-called squashed solution with torsion, was obtained in 1980's using the covariantly constant spinors of the squashed sphere without torsion [5, 10]. We can also take  $\lambda = -1/\sqrt{5}$ ,  $\alpha = -3/\sqrt{5}$ ,  $\beta = -3$ , and  $N = -3R^3/(4\sqrt{5})$  instead, this is the skew-whiffed squashed solution.

### 3 $\mathbf{CP}^3$ as an $S^2$ bundle over $S^4$

In the previous section the metric of  $S^7$  was written as an  $S^3$  bundle over  $S^4$ . It is also possible to write the metric as a  $U(1)$  bundle over  $\mathbf{CP}^3$ . On the other hand, it is observed that  $\mathbf{CP}^3$  itself can be written as an  $S^2$  bundle over  $S^4$ . In this form one can construct a family of homogeneous metrics by rescaling the fibers. In fact, we can see that the metric (3) can be rewritten as a  $U(1)$  bundle over such a deformed  $\mathbf{CP}^3$  [16, 17]. First note that<sup>2</sup>

$$\begin{aligned} ds_{S^7}^2 &= d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (\sigma_i - \cos^2 \mu/2 \Sigma_i)^2 \\ &= d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (d\tau - A)^2 + \lambda^2 (d\theta - \sin \phi A_1 + \cos \phi A_2)^2 \\ &\quad + \lambda^2 \sin^2 \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3)^2, \end{aligned} \quad (29)$$

where

$$A_i = \cos^2 \mu/2 \Sigma_i, \quad (30)$$

and,

$$A = \cos \theta d\phi + \sin \theta (\cos \phi A_1 + \sin \phi A_2) + \cos \theta A_3. \quad (31)$$

$\sigma_i$ 's are left-invariant one-forms that are chosen as follows:

$$\begin{aligned} \sigma_1 &= \sin \phi d\theta + \sin \theta \cos \phi d\tau, \\ \sigma_2 &= -\cos \phi d\theta + \sin \theta \sin \phi d\tau, \\ \sigma_3 &= -d\phi + \cos \theta d\tau. \end{aligned}$$

In the new form of the metric (29), we can further rescale the  $U(1)$  fibers so that the Ricci tensor (in a basis we introduce shortly) is still diagonal. Hence, we take the metric to be

$$\begin{aligned} ds_{S^7}^2 &= d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (d\theta - \sin \phi A_1 + \cos \phi A_2)^2 \\ &\quad + \lambda^2 \sin^2 \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3)^2 + \tilde{\lambda}^2 (d\tau - A)^2, \end{aligned} \quad (32)$$

and choose the following basis

$$\begin{aligned} e^0 &= d\mu, \quad e^i = \frac{1}{2} \sin \mu \Sigma_i, \\ e^5 &= \lambda (d\theta - \sin \phi A_1 + \cos \phi A_2), \\ e^6 &= \lambda \sin \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3), \\ e^7 &= \tilde{\lambda} (d\tau - A). \end{aligned} \quad (33)$$

In this basis the Ricci tensor is diagonal and reads

$$\begin{aligned} R_{00} &= R_{11} = R_{22} = R_{33} = 3 - \lambda^2 - \tilde{\lambda}^2/2, \\ R_{55} &= R_{66} = \lambda^2 + 1/\lambda^2 - \tilde{\lambda}^2/2\lambda^4, \quad R_{77} = \tilde{\lambda}^2 + \tilde{\lambda}^2/2\lambda^4. \end{aligned} \quad (34)$$

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<sup>2</sup>For the sake of clarity, we will set  $R^2 = 4$  from now on.

### 3.1 The ansatz

As in the previous section, a natural 3-form to begin with is  $\omega_3 = e^{567}$ . To proceed, however, it proves useful to define the following forms

$$\begin{aligned} R_1 &= \sin \phi (e^{01} + e^{23}) - \cos \phi (e^{02} + e^{31}), \\ R_2 &= \cos \theta \cos \phi (e^{01} + e^{23}) + \cos \theta \sin \phi (e^{02} + e^{31}) - \sin \theta (e^{03} + e^{12}), \\ K &= \sin \theta \cos \phi (e^{01} + e^{23}) + \sin \theta \sin \phi (e^{02} + e^{31}) + \cos \theta (e^{03} + e^{12}). \end{aligned} \quad (35)$$

The key feature of this definition, that we will use frequently in this paper, is that these three forms are orthogonal to each other, i.e.,

$$R_1 \wedge R_2 = K \wedge R_1 = K \wedge R_2 = 0. \quad (36)$$

Let us also define,

$$\text{Re } \Omega = R_1 \wedge e^5 + R_2 \wedge e^6, \quad \text{Im } \Omega = R_1 \wedge e^6 - R_2 \wedge e^5, \quad (37)$$

we will further need to work out the exterior derivatives of the above forms

$$d\text{Re } \Omega = 4\lambda\omega_4 - \frac{2}{\lambda} e^{56} \wedge K, \quad d\text{Im } \Omega = 0, \quad (38)$$

for  $d\omega_3$  in the new basis we get

$$d\omega_3 = \lambda \text{Im } \Omega \wedge e^7 - \tilde{\lambda} e^{56} \wedge F, \quad (39)$$

with

$$F = dA = -K - e^{56}/\lambda^2. \quad (40)$$

Note that since

$$d\text{Im } \Omega = 0, \quad \text{Im } \Omega \wedge F = -\text{Im } \Omega \wedge K = 0, \quad (41)$$

we have three independent 4-forms  $\omega_4$ ,  $e^7 \wedge \text{Im } \Omega$ , and  $e^{56} \wedge K$ , which are closed and do not contract into each other. Furthermore, the set of these 4-forms is closed under  $d^*$  operation, and hence a suitable ansatz for  $F_4$  is as follows

$$F_4 = N\epsilon_4 + \alpha\omega_4 + \beta e^7 \wedge \text{Im } \Omega + \gamma K \wedge e^{56}, \quad (42)$$

for  $\alpha$ ,  $\beta$ , and  $\gamma$  three real constants. Taking the Hodge dual we have

$$*_1 F_4 = N\omega_3 \wedge \omega_4 + \epsilon_4 \wedge (\alpha\omega_3 - \beta \text{Re } \Omega + \gamma K \wedge e^7). \quad (43)$$

Using

$$de^{56} = \lambda \text{Im } \Omega, \quad dK = -\frac{1}{\lambda} \text{Im } \Omega, \quad (44)$$

and (38), we see that Maxwell equations (17) reduce to

$$\begin{aligned} -\alpha\lambda^2 + N\lambda\beta + \gamma &= 0, \\ \alpha\tilde{\lambda} + 2\beta/\lambda + (\tilde{\lambda}/\lambda^2 + N)\gamma &= 0, \\ N\alpha - 4\lambda\beta + 2\tilde{\lambda}\gamma &= 0. \end{aligned} \tag{45}$$

As for the Einstein equations, we use (34) and the ansatz (42) to obtain

$$\begin{aligned} 3 - \lambda^2 - \frac{\tilde{\lambda}^2}{2} &= \frac{1}{3}(\alpha^2 + \beta^2 + \frac{1}{2}\gamma^2 + \frac{1}{2}N^2), \\ \lambda^2 + \frac{1}{\lambda^2} - \frac{\tilde{\lambda}^2}{2\lambda^4} &= \frac{1}{3}(-\frac{\alpha^2}{2} + \beta^2 + 2\gamma^2 + \frac{1}{2}N^2), \\ \tilde{\lambda}^2 + \frac{\tilde{\lambda}^2}{2\lambda^4} &= \frac{1}{3}(-\frac{\alpha^2}{2} + 4\beta^2 - \gamma^2 + \frac{1}{2}N^2). \end{aligned} \tag{46}$$

In general, it is not easy to solve set of coupled equations (45) and (46). In fact, apart from the known solutions, we have found no real (i.e., real coefficients for  $F_4$ ) solutions. In especial cases, though, we can reduce the equations further and find solutions. Let us start by assuming

$$\lambda = \tilde{\lambda},$$

then by the Einstein equations we must have  $\beta^2 = \gamma^2$ . Taking  $\beta = -\gamma$  yields  $\lambda = \tilde{\lambda} = 1/\sqrt{5}$ ,  $N = -6/\sqrt{5}$ , and  $\alpha^2 = \beta^2 = \gamma^2 = 9/5$  which is the squashed solution (with torsion) of the previous section with  $R_{\mu\nu} = -45/10 g_{\mu\nu}$ .

For  $\beta = \gamma$ , we get  $\lambda = \tilde{\lambda} = 1$ ,  $N = -2$ , and  $\alpha^2 = \beta^2 = \gamma^2 = 1$ ; this is an Englert type solution with  $R_{\mu\nu} = -5/2 g_{\mu\nu}$ . This has the same four-dimensional Ricci tensor as the original solution found by Englert in [2] using parallelizing torsions on the 7-sphere, and later by [6] and [11] using Killing spinors.

### 3.2 Pope-Warner solution

In this section we rederive the Pope-Warner ansatz and the solution [11] using the canonical forms language. Let us then begin by defining

$$\text{Re } L = -R_1 \wedge e^5 + R_2 \wedge e^6, \quad \text{Im } L = R_1 \wedge e^6 + R_2 \wedge e^5. \tag{47}$$

We note that in the vielbein basis (33),  $A$  in (31) can be written as

$$A = \cot \theta \frac{e^6}{\lambda} + \frac{\cot \mu/2}{\sin \theta} (\cos \phi e^1 + \sin \phi e^2), \tag{48}$$

which, together with (35), allows us to write  $de^5$  and  $de^6$  more compactly as

$$de^5 = -e^6 \wedge A + \lambda R_1, \quad de^6 = e^5 \wedge A + \lambda R_2. \tag{49}$$



Taking the exterior derivative once more yields

$$\lambda dR_1 = \lambda R_2 \wedge A + e^6 \wedge K, \quad \lambda dR_2 = -\lambda R_1 \wedge A - e^5 \wedge K. \quad (50)$$

Having derived (49) and (50), it is now easy to prove that

$$d\text{Re } L = -2A \wedge \text{Im } L, \quad d\text{Im } L = 2A \wedge \text{Re } L. \quad (51)$$

To absorb  $A$  into  $e^7$  in the above equations, we define

$$P = e^{-2i\tau} L, \quad (52)$$

by using eqs. (51), we see that

$$dP = -\frac{2i}{\tilde{\lambda}} e^7 \wedge P. \quad (53)$$

On the other hand, note that

$$*L = iL \wedge e^7, \quad (54)$$

so we can write (53) as

$$dP = \frac{2}{\tilde{\lambda}} *P. \quad (55)$$

This implies that for the 4-form field strength we can take

$$F_4 = N\epsilon_4 + \eta e^7 \wedge (\sin 2\tau \text{Re } L - \cos 2\tau \text{Im } L), \quad (56)$$

with  $\eta$  a real constant. Maxwell eq. (17) then requires  $N = -2/\tilde{\lambda}$ , whereas, the Einstein equations imply  $\lambda^2 = 1$ , and  $\tilde{\lambda}^2 = 2$ , together with  $\eta^2 = 2$ . Note that in this solution the  $U(1)$  fibers of  $S^7$  are stretched by a factor of 2.

We can construct another consistent ansatz by taking a linear combination of Pope-Warner ansatz and the one introduced in the previous section. However, by this we get non-zero off diagonal components of energy-momentum tensor, i.e.,  $T_{56} \neq 0$ , unless we set  $\beta = 0$ . Let us then set

$$F_4 = N\epsilon_4 + \alpha \omega_4 + \gamma K \wedge e^{56} + \eta e^7 \wedge (\sin 2\tau \text{Re } L - \cos 2\tau \text{Im } L), \quad (57)$$

Maxwell eqs. (17) and (45) then require

$$N = -2/\tilde{\lambda}, \quad \lambda^2 = \tilde{\lambda}^2 = 1, \quad \alpha = \gamma, \quad (58)$$

while, the Einstein equations imply

$$\alpha^2 = \gamma^2 = \eta^2 = 1, \quad (59)$$

which is the Englert solution with  $R_{\mu\nu} = -5/2 g_{\mu\nu}$ . Note that here we have  $\alpha = \gamma = 1$ , hence the second and the third terms in (57) combine to

$$\omega_4 + K \wedge e^{56} = \frac{1}{2} F \wedge F, \quad (60)$$

with  $F$  the Kähler form defined in (40). We can now recognize (57) as exactly the Englert solution of [11]. The  $F \wedge F$  term and the term proportional to  $\eta$  are each invariant under an  $SU(4)$  symmetry, but with the given values of the constant coefficients,  $\alpha, \beta$ , and  $\gamma$ , the symmetry enhances to  $SO(7)$ .

## 4 A new $AdS_5 \times \mathbf{CP}^3$ compactification

With the ansatz introduced in Sec. 3.1, we can think of eleven dimensional metrics which are direct product of 5 and 6-dimensional spaces with  $F_4$  given by (42) setting  $N$  and  $\beta$  equal to zero. By this, apart from the result of [18] we derive a new solution of  $AdS_5 \times \mathbf{CP}^3$  so that the  $\mathbf{CP}^3$  factor is stretched along its  $S^2$  fiber by a factor of 2.

Let us then take the eleven dimensional spacetime to be the direct product of a 5 and 6-dimensional spaces,

$$ds_{11}^2 = ds_5^2 + ds_6^2. \quad (61)$$

For the 6-dimensional space we take the same metric that appeared in  $S^7$  description in (29):

$$\begin{aligned} ds_6^2 = & d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (d\theta - \sin \phi A_1 + \cos \phi A_2)^2 \\ & + \lambda^2 \sin^2 \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3)^2, \end{aligned} \quad (62)$$

as mentioned before, this is an  $S^2$  bundle over  $S^4$ , and for  $\lambda^2 = 1$  we get the Fubini-Study metric on  $\mathbf{CP}^3$ . By taking the basis  $e^0, \dots, e^6$  as in (33) the Ricci tensor reads

$$\begin{aligned} R_{00} = R_{11} = R_{22} = R_{33} &= 3 - \lambda^2, \\ R_{55} = R_{66} &= \lambda^2 + 1/\lambda^2. \end{aligned} \quad (63)$$

As for  $F_4$ , we choose the following ansatz

$$F_4 = \alpha \omega_4 + \gamma K \wedge e^{56}, \quad (64)$$

which is closed. Taking the Hodge dual we have

$$*_1 F_4 = \epsilon_5 \wedge (\alpha e^{56} + \gamma K). \quad (65)$$

As  $F_4 \wedge F_4 = 0$ , in this case the Maxwell equation reads

$$d *_1 F_4 = -(\alpha \lambda - \gamma / \lambda) \epsilon_5 \wedge \text{Im } \Omega = 0, \quad (66)$$

where use has been made of (44). So, we must have

$$\alpha \lambda^2 = \gamma. \quad (67)$$

The Einstein equations along compact 6 dimensions, on the other hand, imply

$$\begin{aligned} 3 - \lambda^2 &= \frac{1}{3}(\alpha^2 + \frac{1}{2}\gamma^2) = \frac{1}{3}(1 + \frac{\lambda^4}{2})\alpha^2, \\ \lambda^2 + \frac{1}{\lambda^2} &= \frac{1}{3}(-\frac{\alpha^2}{2} + 2\gamma^2) = \frac{1}{3}(-\frac{1}{2} + 2\lambda^4)\alpha^2, \end{aligned} \quad (68)$$

where we used (67) in the last equalities. From the above equations we get two solutions:

$$\lambda^2 = 1, \quad \alpha^2 = \gamma^2 = 4, \quad (69)$$

for which the metric is the standard Fubini-Study metric of  $\mathbf{CP}^3$ . The 5d Ricci tensor becomes

$$R_{\mu\nu} = -2 g_{\mu\nu}, \quad (70)$$

with  $\mu, \nu = 0, \dots, 4$ . Therefore the 5-dimensional spacetime is anti-de Sitter. This solution was first derived in [18].

For the second solution we have

$$\lambda^2 = 2, \quad \alpha^2 = 1, \quad \gamma^2 = 4, \quad (71)$$

with the 5d Ricci tensor;

$$R_{\mu\nu} = -\frac{3}{2} g_{\mu\nu}. \quad (72)$$

This new solution corresponds to an stretched  $\mathbf{CP}^3$ , in which the  $S^2$  fibers are stretched by a factor of 2. Note that, for this solution the 6-dimensional metric is no longer Einstein. Also, note that according to our discussion at the end of the previous section the first solution, (69), has an  $SU(4)$  symmetry, whereas in the new solution, (71), this symmetry is reduced to  $SO(3) \times SO(5)$ , i.e, to the direct product of the symmetry subgroups of the fiber and the base.

## 5 $AdS_2 \times H^2 \times S^7$ compactification

With metric (32) for the  $S^7$ , we can take yet another ansatz for the metric and  $F_4$  and come up with a new compactification. In fact, in this section we obtain a new solution of type  $AdS_2 \times H^2 \times S^7$ , with  $H^2$  a hyperbolic surface. As we will see, this solution exists only in 11-dimensional space with Euclidean signature, and like the Pope-Warner solution the  $S^7$  metric gets stretched along its  $U(1)$  fibers by a factor of 2.

Let the eleven dimensional spacetime to be the direct product of two 2-dimensional spaces and  $S^7$ ,

$$ds_{11}^2 = ds_A^2 + ds_2^2 + ds_{S^7}^2, \quad (73)$$

where  $ds_{S^7}^2$  is the same as (32). For  $F_4$  we take

$$F_4 = N \epsilon_2^A \wedge \epsilon_2 + \alpha \omega_4 + \beta e^7 \wedge \text{Im } \Omega + \gamma K \wedge e^{56} + \epsilon_2 \wedge (\xi_1 K + \eta_1 e^{56}) + \epsilon_2^A \wedge (\xi_2 K + \eta_2 e^{56}), \quad (74)$$

note that the first four terms are the same as those appeared in (42).  $\xi_1, \xi_2, \eta_1$ , and  $\eta_2$  are constant parameters. We take the 4-dimensional space to be the direct product of two Euclidean subspaces with  $\epsilon_2^A$  and  $\epsilon_2$  as their 2-dimensional volume elements.

The Bianchi identity requires that

$$\xi_1 = \lambda^2 \eta_1, \quad \xi_2 = \lambda^2 \eta_2, \quad (75)$$

For the Maxwell equation, first note that in Euclidean 11-dimensional space we need to account for an extra  $i$  factor coming from the Chern-Simons term so that (17) is replaced by

$$d *_{11} F_4 = -\frac{i}{2} F_4 \wedge F_4, \quad (76)$$

therefore, with our ansätze (32) and (74) the Maxwell equations reduce to the following algebraic equations

$$\begin{aligned} \tilde{\lambda} (2\xi_1 + \eta_1/\lambda^2) &= -i(2\xi_2\gamma + \alpha\eta_2), \\ \tilde{\lambda} (2\xi_2 + \eta_2/\lambda^2) &= -i(2\xi_1\gamma + \alpha\eta_1), \\ \alpha\tilde{\lambda} + 2\beta/\lambda + \gamma(\tilde{\lambda}/\lambda^2 + iN) &= -i(\xi_1\eta_2 + \eta_1\xi_2), \\ iN\alpha - 4\lambda\beta + 2\tilde{\lambda}\gamma &= -2i\xi_1\xi_2, \\ -\alpha\lambda^2 + iN\lambda\beta + \gamma &= 0. \end{aligned} \quad (77)$$

Using (75), the first two equations above imply

$$\xi_1^2 = \xi_2^2, \quad \eta_1^2 = \eta_2^2. \quad (78)$$

Had we chosen a Lorentzian signature metric for the 4-dimensional space, since  $*\epsilon_2^A = -\epsilon_2$  and  $*\epsilon_2 = \epsilon_2^A$  we would have obtained  $\xi_1^2 = -\xi_2^2$ , with no real solution. On the other hand, in Euclidean 11-dimensional space eq. (76) implies that whenever the RHS is nonvanishing  $F_4$  is necessarily complex valued, and so there is no restriction on the coefficients of  $F_4$  to be real. However, for having a well-defined metric we still require that  $\lambda$  and  $\tilde{\lambda}$  to be real.

To carry on, we set  $\xi_1 = \xi_2 \equiv \xi$  without loss of generality, and (77) becomes

$$\begin{aligned} \alpha\lambda^2 + 2\lambda^4\gamma - i\tilde{\lambda}(1 + 2\lambda^4) &= 0, \\ (\tilde{\lambda}\lambda^2 - iN)\alpha + 6\lambda\beta + (iN\lambda^2 - \tilde{\lambda})\gamma &= 0, \\ \tilde{\lambda}\gamma - 2\lambda\beta + iN\alpha/2 + i\xi^2 &= 0, \\ -\alpha\lambda^2 + iN\lambda\beta + \gamma &= 0, \end{aligned} \quad (79)$$

where the second equation is obtained by dividing the third and fourth equations in (77). For  $\beta \neq 0$ , we have found no solution of (79) for which  $\lambda$  and  $\tilde{\lambda}$  are both real. Let us then discuss the case with  $\beta = 0$ . In this case, the second and the forth equations above imply

$$\lambda^2 = 1, \quad \alpha = \gamma. \quad (80)$$

Plugging this into the first equation we have

$$\alpha = \gamma = i\tilde{\lambda}, \quad (81)$$

and finally the third equation gives

$$\xi^2 = -\tilde{\lambda}^2 - \frac{i}{2}\tilde{\lambda}N. \quad (82)$$

Let us now look at the Einstein equations along  $S^7$ . Taking into account  $\lambda^2 = 1$  and  $\alpha = \gamma$ , they read

$$\begin{aligned} 2 - \frac{\tilde{\lambda}^2}{2} &= \frac{1}{2}\alpha^2 - \frac{1}{6}N^2 - \frac{1}{2}\xi^2, \\ \frac{3}{2}\tilde{\lambda}^2 &= -\frac{1}{2}\alpha^2 - \frac{1}{6}N^2 - \frac{1}{2}\xi^2, \end{aligned} \quad (83)$$

note the sign change of  $N^2$  as a result of using the Riemannian signature (compare with (46)). Using (81), we can solve for  $\tilde{\lambda}$ :

$$\tilde{\lambda}^2 = 2, \quad (84)$$

and,

$$N^2 + 3\xi^2 + 12 = 0, \quad (85)$$

this last equation together with (82) can be solved to give  $N$  and  $\xi$ .

The Ricci tensor along two 2-dimensional spaces reads

$$\begin{aligned} R_{ab} &= (\xi_2^2 + \frac{1}{2}\eta_2^2 + \frac{1}{2}N^2 - \frac{1}{6}(N^2 + \alpha^2 + 2\gamma^2) - \frac{1}{2}\xi^2) g_{ab}, \\ R_{a'b'} &= (\xi_1^2 + \frac{1}{2}\eta_1^2 + \frac{1}{2}N^2 - \frac{1}{6}(N^2 + \alpha^2 + 2\gamma^2) - \frac{1}{2}\xi^2) g_{a'b'}, \end{aligned} \quad (86)$$

with  $a, b = 0, 1$ , and  $a', b' = 2, 3$ . Now, using (81), (84), and (85), we get

$$\begin{aligned} R_{ab} &= -3 g_{ab}, \\ R_{a'b'} &= -3 g_{a'b'}. \end{aligned} \quad (87)$$

Therefore, the 4-dimensional space is a direct product of a Euclidean  $AdS_2$  and a 2-dimensional hyperbolic surface. Interestingly, this solution has some common features with the Pope-Warner and the Freund-Rubin solutions. As in the Pope-Warner solution, here the metric of  $S^7$  is stretched by a factor of 2 along its  $U(1)$  fiber with  $SU(4)$  isometry group. And, on the other hand, the 4-dimensional Ricci tensor is equal to that of the Freund-Rubin solution.

## 6 Conclusions

In this paper we provided a unified approach to study the squashed, stretched, and the Englert type solutions of 11-dimensional supergravity, especially when there are fluxes in the compact direction. With the special form of the metric (32), we

were able to construct more general ansätze by bringing together the earlier known ones and those constructed in Sec. 3. We then used the ansatz to reduce the field equations to algebraic ones and rederive the known solutions. Further, using these ansätze we were able to find new compactifying solutions to 5 and 4 dimensions. In compactifying to 5 dimensions, we derived a solution of  $AdS_5 \times \mathbf{CP}^3$  type with the  $\mathbf{CP}^3$  factor stretched. We also derived a solution of  $AdS_2 \times H^2 \times S^7$  type compactifying to Euclidean 4 dimensions. In this solution, the compact space was a stretched  $S^7$ .

Having derived the above solutions, the next important issue to address is that of stability. It is also worth studying the new solutions in the context of holographic superconductivity in M-theory [19].

## References

- [1] P. G. O. Freund and M. A. Rubin, *Dynamics Of Dimensional Reduction*, Phys. Lett. B **97**, 233 (1980).
- [2] F. Englert, *Spontaneous Compactification Of Eleven-Dimensional Supergravity*, Phys. Lett. B **119**, 339 (1982).
- [3] M. A. Awada, M. J. Duff and C. N. Pope,  *$N = 8$  supergravity breaks down to  $N = 1$* , Phys. Rev. Lett. **50**, 294 (1983).
- [4] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Spontaneous Supersymmetry Breaking By The Squashed Seven Sphere*, Phys. Rev. Lett. **50**, 2043 (1983).
- [5] F. A. Bais, H. Nicolai and P. van Nieuwenhuizen, *Geometry Of Coset Spaces And Massless Modes Of The Squashed Seven Sphere In Supergravity*, Nucl. Phys. B **228**, 333 (1983).
- [6] M. J. Duff, *Supergravity, The Seven Sphere, And Spontaneous Symmetry Breaking*, Nucl. Phys. B **219**, 389 (1983).
- [7] L. Castellani and L. J. Romans,  *$N=3$  And  $N=1$  Supersymmetry In A New Class Of Solutions For  $D = 11$  Supergravity*, Nucl. Phys. B **238**, 683 (1984).
- [8] D. V. Volkov, D. P. Sorokin and V. I. Tkach, *Supersymmetry Vacuum Configurations in  $D = 11$  Supergravity*, JETP Lett. **40**, 1162 (1984).
- [9] D. P. Sorokin, V. I. Tkach and D. V. Volkov, *On the Relationship between Compactified Vacua of  $D = 11$  and  $D = 10$  Supergravities*, Phys. Lett. B **161**, 301 (1985).
- [10] F. Englert, M. Roman and P. Spindel, *Supersymmetry Breaking By Torsion And The Ricci Flat Squashed Seven Spheres*, Phys. Lett. B **127**, 47 (1983).

- [11] C. N. Pope and N. P. Warner, *An  $SU(4)$  Invariant Compactification Of  $D = 11$  Supergravity On A Stretched Seven Sphere*, Phys. Lett. B **150**, 352 (1985).
- [12] V. D. Lyakhovsky and D. V. Vassilevich, *Algebraic Approach to Kaluza-Klein Models*, Lett. Math. Phys. **17**, 109 (1989).
- [13] J. P. Gauntlett, S. Kim, O. Varela and D. Waldram, *Consistent supersymmetric Kaluza-Klein truncations with massive modes*, JHEP **0904**, 102 (2009) [arXiv:0901.0676 [hep-th]].
- [14] D. Cassani and P. Koerber, *Tri-Sasakian consistent reduction*, JHEP **1201**, 086 (2012) [arXiv:1110.5327 [hep-th]].
- [15] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein Supergravity*, Phys. Rept. **130**, 1 (1986).
- [16] B. E. W. Nilsson and C. N. Pope, *Hopf Fibration Of Eleven-Dimensional Supergravity*, Class. Quant. Grav. **1**, 499 (1984).
- [17] G. Aldazabal and A. Font, *A Second look at  $N=1$  supersymmetric  $AdS(4)$  vacua of type IIA supergravity*, JHEP **0802**, 086 (2008), [arXiv:0712.1021].
- [18] C. N. Pope and P. van Nieuwenhuizen, *Compactifications of  $d = 11$  Supergravity on Kähler Manifolds*, Commun. Math. Phys. **122**, 281 (1989).
- [19] J. P. Gauntlett, J. Sonner and T. Wiseman, *Quantum Criticality and Holographic Superconductors in M-theory*, JHEP **1002**, 060 (2010), [arXiv:0912.0512 [hep-th]].